Quantum Simulation of Dihedral Gauge Theories

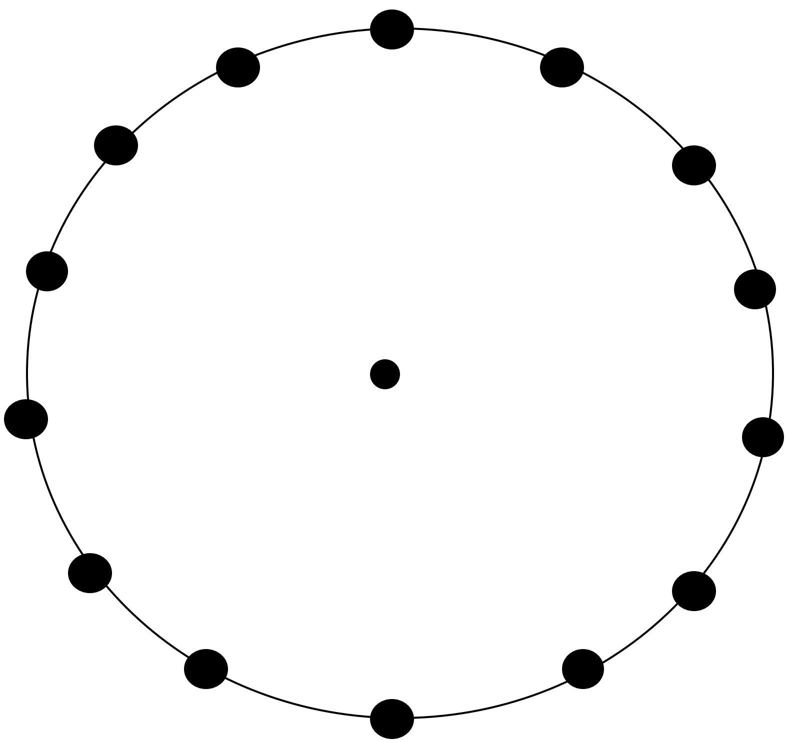
M. Sohaib Alam

NASA QuAIL, Ames Research Center

APS March Meeting 2022

Based on arXiv: 2108.13305 in collaboration with Stuart Hadfield (NASA), Henry Lamm (FermiLab), Andy C. Y. Li (Fermilab)

General Idea



Approximate a continuous group by a discrete subgroup

$$U(1) \approx \mathbb{Z}_N$$

Simple example of a non-Abelian discrete group

$$D_N \sim \mathbb{Z}_N \rtimes \mathbb{Z}_2$$

Generic element

$$s^m r^k$$

$$m \in \{0, 1\}\,, \quad z \in \{0, \ldots, N-1\}$$

General Methods

- **For some generic gauge group *G*,**
  - Define a *G*-register with one basis element $|g\rangle$ per group element $g \in G$
  (Hilbert space on single *G*-register $\mathcal{H}_G = \mathbb{C}G$; for *L* links on entire lattice, $\mathcal{H} = \mathbb{C}G^{\otimes L}$)
- **Define basic gates**
  - Inverse gate
  $$\mathcal{U}_{-1}|g\rangle = |g^{-1}\rangle$$
  - (Left) Multiplication gate
  $$\mathcal{U}_{\times}|g\rangle|h\rangle = |g\rangle|gh\rangle$$
  - Trace gate
  $$\mathcal{U}_{\mathrm{Tr}}(\theta)|g\rangle = e^{i\theta \,\mathrm{Re\ Tr\ g}}|g\rangle$$
  - Fourier gate
  $$\mathcal{U}_F(\theta)\sum_{g\in G} f(g)|g\rangle = \sum_{\rho\in\hat{G}} \hat{f}(\rho)_{ij}|\rho, i, j\rangle$$

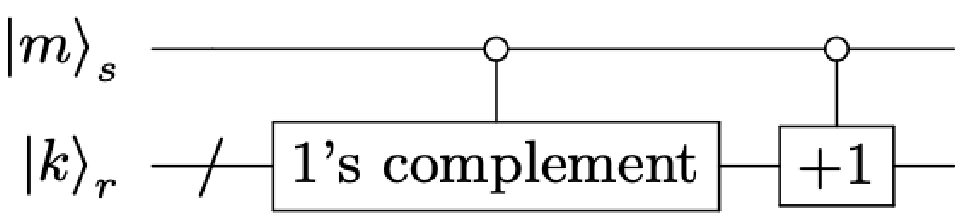
"General methods for digital quantum simulation of gauge theories", H. Lamm et al., Phys Rev D

Inverse Gate

Computing the 2's complement of k

$$\left(s^m r^k\right)^{-1} = s^m r^{Nm+(-1)^m k}$$

Equivalent to computing 1's complement, then adding 1

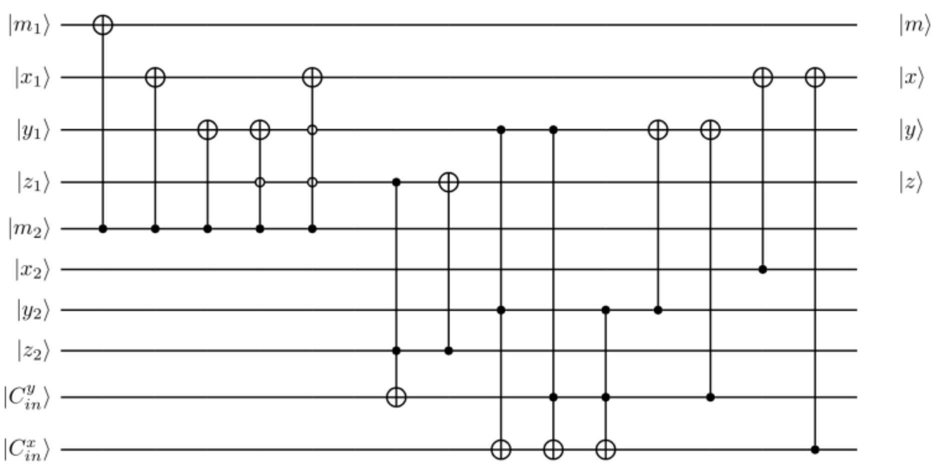


n CNOTs

O(n) CCNOTs + const # ancillas

Multiplication Gate

$$s^{m_1} r^{k_1} \cdot s^{m_2} r^{k_2} = s^{m_1+m_2} r^{N m_2 + (-1)^{m_2} k_1 + k_2}$$



Example circuit for D8 multiplication gate

$m_2 = 0:$ add k_1 and k_2

$m_2 = 1:$ add 2's complement of k_1 and k_2

Sum and Carry bits in Reed-Muller form

$$S = A \oplus B \oplus C_{in}$$
$$C_{out} = AB \oplus AC_{in} \oplus BC_{in}$$

Trace Gate

Identify corresponding Hamiltonian

$$U_{\mathrm{Tr}}(\theta)\,|g\rangle = e^{i\theta\,\mathrm{Re}(\mathrm{Tr}(g))} = e^{i\theta H_{\mathrm{Tr}}} \qquad H_{\mathrm{Tr}}\,|g\rangle = \mathrm{Re}(\mathrm{Tr}(g))\,|g\rangle$$

Fundamental (2D) representation

$$\rho(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}^k \qquad \text{where} \quad \omega = e^{2\pi i/N}$$

$$\Rightarrow \qquad H_{\mathrm{Tr}} = |0\rangle\langle 0| \otimes \sum_{\ell=0}^{N-1} 2\cos(2\pi\ell/N)\,|\ell\rangle\langle\ell|$$

O(N) implementation: $U_{\mathrm{Tr}}(\theta) = \prod_{j=0}^{N} e^{i\theta a_\alpha |0\rangle\langle 0|\otimes Z_\alpha}$

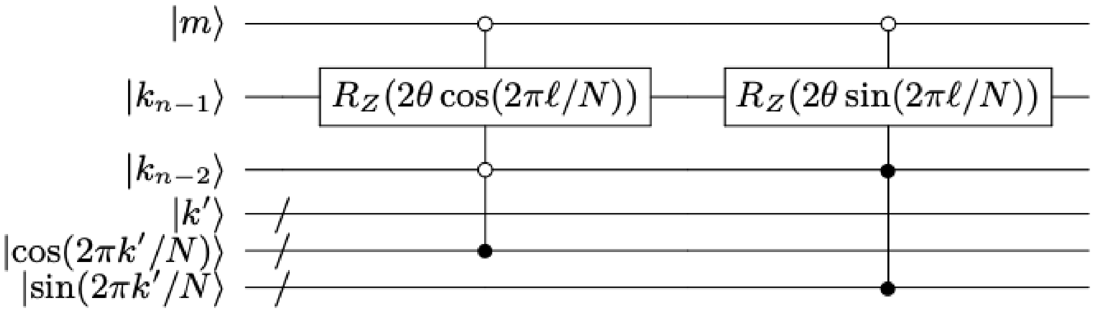
Trace Gate

Ancilla Based Approach (*O(log(N))*)

- **Compute (classical) trigonometric functions**

$$|mk_{n-1}k_{n-2}k'\rangle\,|0\ldots0\rangle \rightarrow |mk_{n-1}k_{n-2}k'\rangle\,|\widetilde{\sin(2\pi k'/N)}\rangle\,|\widetilde{\cos(2\pi k'/N)}\rangle\,|scratchpad\rangle$$

- **Phase Kickback**



- **Uncomputation**

$$|g\rangle = |mk\rangle \rightarrow e^{i\theta 2(1-m)\cos(2\pi\ell_b/N)}\,|mk\rangle = e^{i\theta Tr(g)}\,|g\rangle$$

Fourier Gate

Fourier transform of a representation of some finite group G

$$\hat{f}(\rho) = \sqrt{\frac{d_\rho}{N}} \sum_{g \in G} f(g)\rho(g)$$

Recursive definition

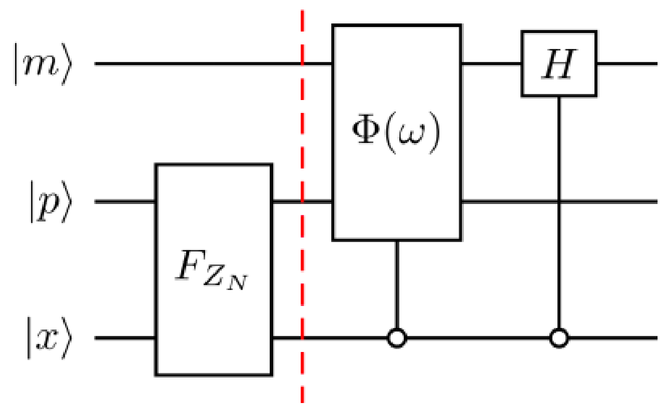
$$\sum_{g \in G} f(g)\rho(g) = \sum_{i=1}^{n} \rho(g_i)\hat{f}_i(\rho|_H)$$

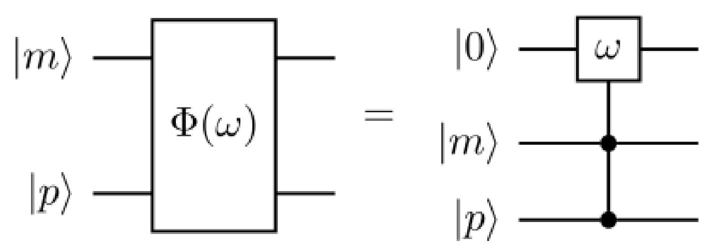
Fourier Gate

$$\sum_{g\in G} \alpha_g |g\rangle \;=\; \sum_{i=1} \sum_{h\in H} \alpha(g_i h)|g_i\rangle|h\rangle$$

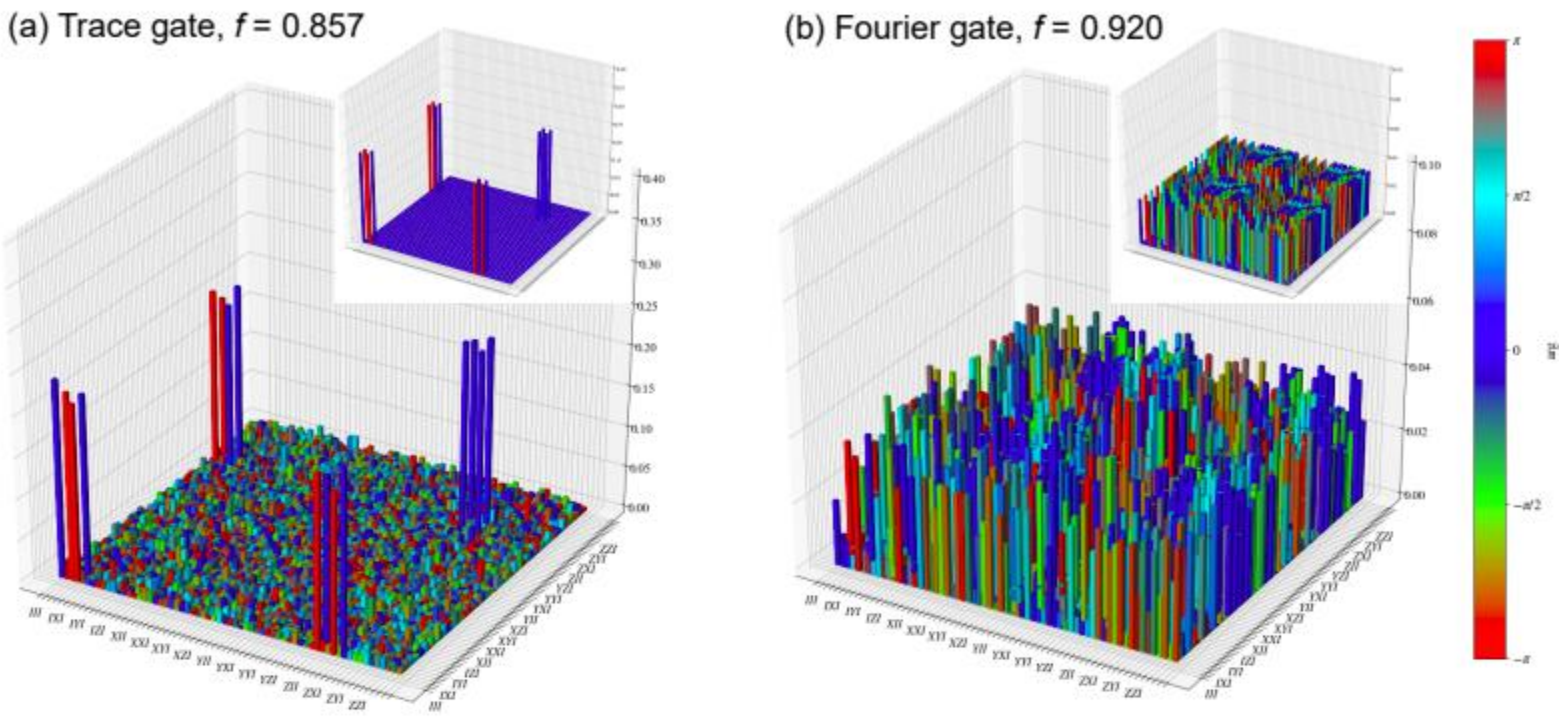
$$\xrightarrow{F_H} \sum_{i=1}^{n} |g_i\rangle \left(\sum_{\tilde{h}\in\hat{H}} \hat{\alpha}_i(\tilde{h})|\tilde{h}\rangle \right)$$

$$\xrightarrow{U} \sum_{\tilde{g}\in\hat{G}} \hat{\alpha}(\tilde{g})|\tilde{g}\rangle$$



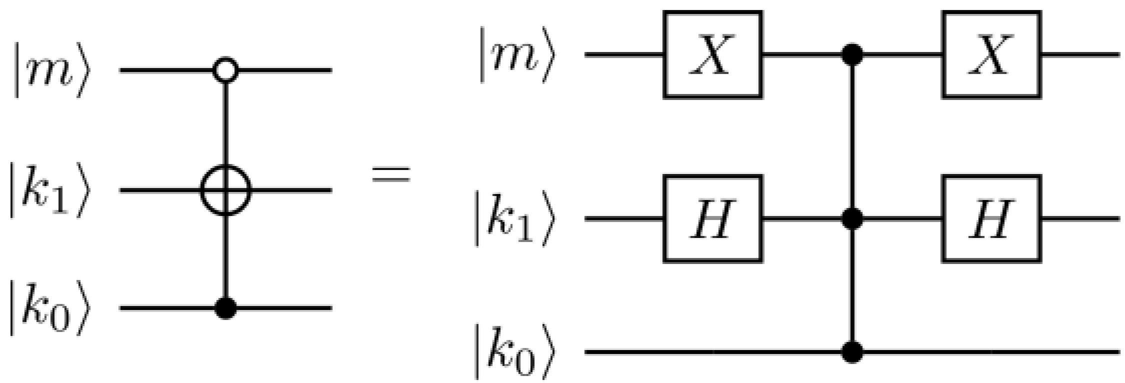


Experimental Results – Rigetti Aspen-9



Experimental Results – Rigetti Aspen-9

(c) Inversion gate for D4



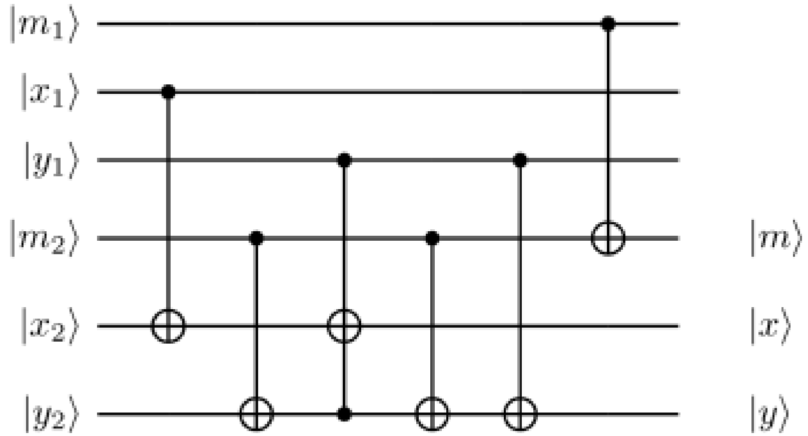
Process fidelity of CCPHASE gate ~ 87.1% using cycle benchmarking

"Realization of arbitrary doubly-controlled quantum phase gate", A. Hill et al., arXiv: 2108.01652

"Characterizing large-scale quantum computers via cycle benchmarking", A. Erhard et al., Nature Communications

Experimental Results – Rigetti Aspen-9

(d) Multiplication gate for D4



Average fraction of correct output bitstrings $\sim 0.89\,(\Delta \sim 0.18)$

Majority vote of 200 successive shots, with a total of 10,000 shots $\sim 0.91\,(\Delta \sim 0.15)$

Thank you!